Model-Based synthesis of indicators
Statistical Composite Indicators to convey consistent policy messages

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Workshop: THE IMPACTS AND METHODOLOGY OF INDICATORS AND SCOREBOARDS
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Outline of the Presentation

FROM DATA to KNOWLEDGE by Dimensionality Reduction:
a model-based Composite Indicator (CI) is the result of the dimensional reduction of the observed multivariate data.

We will start from

• Properties on which a Composite Indicator should be based

We will discuss

• HCI : Hierarchical Composite Indicator Model

• Application study:
  • Human Development Index;
  • Multidimensional Poverty Index
  • SDGs for Europe with data by EUROSTAT
From DATA to KNOWLEDGE via Data Dimensionality Reduction for DECISION MAKING

DATA

GENERAL COMPOSITE INDICATOR

SPECIFIC COMPOSITE INDICATORS

INFORMATION

Heat-map

INDICATOR

Indicator 1 Indicator 2 Indicator 3 Indicator 4 Indicator 5 Indicator 6 Indicator 7 Indicator 8
Moving from DATA to KNOWLEDGE by a Dimensionality Reduction
Let us start from the **Observed Data (Manifest Indicators)** on the multidimensional concept formed by the observed set of indicators.
Specific Composite Indicators for detecting the “specific concepts” describing the phenomenon

First Level Synthesis

Each SCI synthesises a set of indicators

First Order Specific Composite Indicators

Each SCI synthesises a set of indicators
Specific Composite Indicators for detecting the “main concepts” describing the phenomenon (2/2)

Second Level Synthesis

Each SCI synthesises a set of indicators

Second Order Specific Composite Indicators

Each SCI synthesises a set of indicators or factors
General Composite Indicator (for decision making)

Third Level Synthesis

Hierarchical CI model to move from DATA to KNOWLEDGE
PROPERTIES of CI

FIRST PART
  Model-Based &
  Statistically estimated (non-normative)
    Confirmatory, Exploratory or Mixed
    Reflective and/or Formative
  Model assessment

SECOND PART
  Scale-invariance
  Non-Compensability & Non-Negativity
  Reliability
  Unidimensionality, forming a General CI
PROPERTIES of CI

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Statistical Model: Hierarchical CI

Model-based CI & its statistical estimation (i.e., non-normative):

Data = Hierarchical CI model + error

Advantages
Statistical estimation (LS, MLE, …)
Validation: Goodness of Fit (to confirm the model)
Inference on the weights, GoF, …

Which typology of constructive approach:
- **Confirmatory** – a Scientific Theory (ST) is assumed and has to be confirmed by the observed indicators;
- **Exploratory** – no clear ST is known, thus, regularities are searched in the data;
- **Mixed Confirmatory & Exploratory** – part of the ST is known, but it is not completely known

Which typology of relations between indicators:
- **Reflective**
- **Formative**
Confirmatory, Exploratory, Mixed-Confirmatory/Exploratory

- **Confirmatory** model: if a theory on the model of the CI is available, i.e., all relationships between manifest variables and latent variables are and a priori known;
- **Exploratory** model: all relationships between manifest variables and latent variables are not a priori known;
- **Mixed-confirmatory/exploratory** : some relationships are known according to a theory and some are unknown and must be achieved by exploratory analysis.
Relations between Composite Indicators (GCI & SCIs) and Manifest Indicators

A) Reflective
B) Formative

The General Composite Indicator is a determinant (causes) the Specific Composite Indicators & these last are determinant (causes) of the Manifest Indicators, i.e., The GCI reconstructs the SCIs that reconstruct the MI

Independent Manifest Indicators are determinant (cause, explain) of independent Specific Composite indicators that are determinant of the General Composite Indicator
Hierarchical Composite Indicator (HCl)

A model to identify the latent Hierarchical Composite Indicator and the set of specific Composite Indicators that best reconstruct the observed data.

**SOME METODOLOGICAL CONSIDERATIONS**

\[
X = g c' V' B + E
\]

Parameters of the model:
- \( c \): weights for SCIs
- \( V \): relations between MIs & SCIs

\[
\hat{V}' = \begin{pmatrix}
1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1
\end{pmatrix}
\]

Weights for variables:
- \( \hat{B} = diag(b) \)
- \( b_1, b_2, \ldots, b_{10} \)
Special cases of HDFA\textsubscript{(1/2)}

g=arithmetic mean of MIs if : \(c_1 = c_2 = \ldots = c_Q = 1; b_1 = b_2 = \ldots b_J = 1 \) (equal weights)

\[
\hat{g}_M = X(1_H \hat{V}')^+ = X1_J^+ = \frac{1}{J} (x_1 + x_2 + \ldots + x_J),
\]

\[
X = gc' V' B + E
\]
Special cases of HCl (2/2)

\( g = \text{weighted arithmetic mean of MIs} \) (i.e., different weights)

\[ \hat{g}_{WM} = X(\hat{B}\hat{V}\hat{c})(\hat{c}^\prime\hat{V}'\hat{B}1_j)^{-1} \]

Data \hspace{1cm} MODEL \hspace{1cm} ERROR

\[ X=gc'V'B+E \]

Parameters of the model
\( c \) weights for SCIs
\( V \) relations between MIs & SCIs

\[ \hat{V}' = \begin{pmatrix} 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \]

Weights for MIs
\( \hat{B} = \text{diag}(b) \)
\( b_1, b_2, \ldots, b_{10} \)
PROPERTIES of CI

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Model assessment

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Reliability
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**MODEL ASSESSMENT**

The **goodness of fit** of the CI model:

\[ R^2_{GCI} = 1 - \frac{SS_{res}}{SS_{tot}} = 1 - \frac{tr(X'X) - tr((B\hat{c}\hat{g})\hat{g}(\hat{c}'\hat{V}'B))}{tr(X'X)} \]

\[ R^2_{SCI} = 1 - \frac{SS_{res}}{SS_{tot}} = 1 - \frac{tr(X'X) - tr(B\hat{V}\hat{Y}'\hat{Y}'B)}{tr(X'X)} \]

\[ R^2_{SCI_h} = 1 - \frac{SS_{resy_h}}{SS_{tot_h}} = 1 - \frac{tr(X_{h}'X_{h}) - tr(\hat{B}_h\hat{v}_h\hat{y}_h'\hat{v}_h\hat{y}_h'\hat{B}_h)}{tr(X_{h}'X_{h})} \]

The **Information criteria**

**AIC**
\[-2\log OI(\theta, \pi) + 2d\]

**BIC**
\[-2\log OI(\theta, \pi) + d \log n\]
Example 1: Assessment of the Model-Based CI Case of ARITHMETIC MEAN

\[ X = gc'V'B + E \]

g \sim N(0, 1)

If \( \hat{c} = 1_3 \) and \( \hat{B} = I_{10} \),

\[ \hat{V}' = \begin{pmatrix} 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \end{pmatrix} \]

<table>
<thead>
<tr>
<th>Error:</th>
<th>( R^2_{GCI} )</th>
<th>( R^2_{SCL_1} )</th>
<th>( R^2_{SCL_2} )</th>
<th>( R^2_{SCL_3} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( X_s ) Small</td>
<td>0.974</td>
<td>0.988</td>
<td>0.988</td>
<td>0.989</td>
</tr>
<tr>
<td>( X_m ) Medium</td>
<td>0.622</td>
<td>0.778</td>
<td>0.837</td>
<td>0.855</td>
</tr>
<tr>
<td>( X_L ) Large</td>
<td>0.131</td>
<td>0.624</td>
<td>0.539</td>
<td>0.672</td>
</tr>
</tbody>
</table>

Arithmetic mean is a good GCI only when the MIs are very similar.
Example 2: Assessment of the Model-Based CI
Case of ARITHMETIC MEAN

\[ R^2_{GCI} = \frac{SS_{mod}}{SS_{tot}} = 0.548 \]

In a situation like this it is better to stop at an intermediate level of synthesis (i.e., SCIs level) because a GCI built as the arithmetic mean of MIs is not a good representation of the phenomenon to describe.
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Example 2: Assessment of the Model-Based CI
Case of ARITHMETIC MEAN

\[ R_{GCI}^2 = \frac{SS_{mod}}{SS_{tot}} = 0.548 \]

<table>
<thead>
<tr>
<th></th>
<th>( R_{SCI1}^2 )</th>
<th>( R_{SCI2}^2 )</th>
<th>( R_{SCI3}^2 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Value</td>
<td>0.999</td>
<td>0.999</td>
<td>1</td>
</tr>
</tbody>
</table>
Example 2: Assessment of the Model-Based CI
Case of ARITHMETIC MEAN

Final correct description

\[ R^2_{GCI} = \frac{SS_{mod}}{SS_{tot}} = 0.548 \]

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Reliability
Unidimensionality, forming a General CI
Scale-invariance

Data are normalized in order to allow the comparison and the combination of the MIs into the SCIs and GCI.

- **Standardization**
  \[ Z = JX \text{diag}(d\Sigma_x)^{-1/2} \quad \text{with} \quad J = I_n - (1/n) 1_n 1_n' \]

- **Min-max normalization**
  \[ Z = X - 1_n \text{min}X. / (1_n \text{max}X - 1_n \text{min}X) \]

- **Normalized dispersion**
  \[ Z = JX \text{diag}(\mu_x)^{-1} \quad \text{with} \quad J = I_n - (1/n) 1_n 1_n' \]

A scale-invariant CI is a latent Indicator that is not sensitive to linear transformations such as normalization methods.
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Non-Compensability & Non-Negativity.
The CI satisfies the non-compensability property if its relationships with latent and/or MIs are all positives. Thus, the effect of the SCIs and/or MIs do not compensate each other.

So non-negativity and non-compensability are strictly connected.
Non-Compensability & Non-Negativity.
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  Reliability
  Unidimensionality, forming a general CI
Reliability, Unidimensionality & General Factor

**Reliability** of a CI is the global consistency of MIs based on the correlations between different MIs on the same CI. It is frequently called internal consistency and it is usually measured with Cronbach’s alpha (Cronbach, 1951).

**Unidimensionality** evaluates to which extend a single latent indicator, generally a SCI, has been measured with a set of MIs. Unidimensionality is more realistic for SCIs, while Revelle and Zinbarg, (2009) hypothesize that there is a general factor, i.e., a GCI that can be tested by nested confirmatory SCIs.

A measure of unidimensionality for each SCI might be the variance of the second component of the the set of MIs explained by the related SCI.

**Example:**

<table>
<thead>
<tr>
<th></th>
<th>Factor 1</th>
<th>Factor 2</th>
</tr>
</thead>
<tbody>
<tr>
<td>Unidimensionality</td>
<td>2.737</td>
<td>0.556</td>
</tr>
<tr>
<td>Reliability</td>
<td>0.526</td>
<td>0.476</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th></th>
<th>Factor 1</th>
<th>Factor 2</th>
<th>Factor 3</th>
</tr>
</thead>
<tbody>
<tr>
<td>Unidimensionality</td>
<td>0.400</td>
<td>0.556</td>
<td>0.618</td>
</tr>
<tr>
<td>Reliability</td>
<td>0.781</td>
<td>0.794</td>
<td>0.781</td>
</tr>
</tbody>
</table>
APPLICATIONS
Human Development Index - HDI

The HDI is the geometric mean of the previous three normalized indices; and we can measure the goodness of fit of the HDI by considering that the logarithm of the geometric mean is equal to the arithmetic mean of the logarithm of MIs. Each dimension is represented by a specific index (normalized with a own method): Life Expectancy Index (LEI), Education Index (EI) and Income Index (II).

Let us consider:

\[ \hat{B} = \hat{V} = I_3 \]

\[ \hat{c} = I_3 \]

\[ R_{HDI}^2 = \frac{SS_{mod}}{SS_{tot}} = \frac{tr(\hat{B}\hat{V}\hat{c} log(\hat{g}_{HDI})' log(\hat{g}_{HDI})\hat{c}' \hat{V}' \hat{B})}{tr((log(\mathbf{X}))'(log(\mathbf{X})))} \]

\[ = 0.901 \]

where \( log(\mathbf{X}) \) is a matrix where each column is the logarithmic transformation of the respectively column of \( \mathbf{X} \).

Based on the above informations:
- Life Expectancy Index (LEI) = Actual LE – 20/(85-20)
- Income Index (II) = \{ln(GNI pc)- ln(100)\}/\{ln(75,000) – ln(100)\}
- Education Index (EI) = MYSI+EYSI / 2
- Mean Years of Schooling Index (MYSI) = MYS-0 / 15-0
- Expected Years of Schooling Index (EYSI) = EYS-0 / 18-0

Now, HDI is the geometric mean of previous three indices i.e.

\[ \text{HDI} = \sqrt[3]{\text{LEI} \times \text{EI} \times \text{II}} \]
**Multidimensional Poverty Index- MPI**

The global Multidimensional Poverty Index (MPI) is an international measure of acute poverty covering over 100 developing countries developed by OPHI and the United Nations Development Programme. The index uses the same three dimensions as the Human Development Index: **health**, **education**, and **standard of living**. These are measured using ten indicators divided in three dimensions.

Let us consider:

\[
\hat{\mathbf{B}} = \text{diag}(\begin{array}{cccccccc}
\frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2}
\end{array})
\]

\[
\hat{\mathbf{V}}' = \begin{pmatrix} 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 
\end{pmatrix}
\]

\[
\hat{\mathbf{c}}' = (\frac{1}{3} \ \frac{1}{3} \ \frac{1}{3})
\]

\[
R^2_{\text{MPI}} = \frac{SS_{\text{mod}}}{SS_{\text{tot}}} = \frac{\text{tr}((\hat{\mathbf{c}}'\hat{\mathbf{V}}'\hat{\mathbf{B}}\hat{\mathbf{V}}\hat{\mathbf{c}})^{-1}(\hat{\mathbf{B}}\hat{\mathbf{V}}\hat{\mathbf{c}})\hat{\mathbf{g}}_{\text{MPI}}'\hat{\mathbf{g}}_{\text{MPI}}(\hat{\mathbf{c}}'\hat{\mathbf{V}}'\hat{\mathbf{B}}\hat{\mathbf{V}}\hat{\mathbf{c}})^{-1})}{\text{tr}(\mathbf{X}'\mathbf{X})} = 0.6667
\]
Application to Sustainable Development Goals
SDGs Europe: 100 Indicators, 17 Goals

Goal 1:
1- People at risk of poverty or social exclusion 01.11
2- People at risk of poverty after social transfers 01.12
3- Severely materially deprived people 01.13
4- People living in households with very low work intensity 01.14
5- Housing cost overburden rate 01.21
6- Share of total population living in a dwelling with a leaking roof, damp walls, floors or foundation, or rot in window frames or floor 01.22

Goal 2:
7- Obesity rate 02.11
8- Agricultural factor income per annual work unit (AWU) 02.21
9- Government support to agricultural research and development 02.26
10- Area under organic farming 02.31
11- Ammonia emissions from agriculture 02.52
12- Gross nutrient balance on agricultural land 02.54

Goal 3:
13- Life expectancy at birth 03.11
14- Self-perceived health 03.14
15- Death rate due to chronic diseases 03.25
16- Suicide death rate 03.31
17- Smoking prevalence 03.36
18- Self-reported unmet need for medical examination and care 03.41

Goal 4:
19- Early childhood education and care 04.10
20- Early leavers from education and training 04.20
21- Tertiary educational attainment 04.30
22- Employment rate of recent graduates 04.31
23- Adult participation in learning 04.40
24- Underachievement in reading, maths and science 04.50

Goal 5:
25- Gender pay gap 05.10
26- Gender employment gap 05.12
27- Proportion of seats held by women in national parliaments and local government 05.20
28- Proportion of women in senior management positions 05.21
29- Physical and sexual violence by a partner or a non-partner 05.33
30- Inactivity rates due to caring responsibilities 05.44

Goal 6:
31- Share of total population having neither a bath, nor a shower, nor indoor flushing toilet in their household 06.11
32- Population connected to urban wastewater treatment with at least secondary treatment 06.13
33- Biochemical oxygen demand in rivers 06.21
34- Nitrate in groundwater 06.24
35- Phosphate in rivers 06.26
36- Water exploitation index (WEI) 06.41

Goal 7:
37- Percentage of people affected by fuel poverty (inability to keep home adequately warm) 07.10
38- Share of renewable energy in gross final energy consumption 07.20
39- Primary energy consumption; final energy consumption by sector 07.30
40- Final energy consumption in households per capita 07.32
41- Energy dependence 07.33
42- Energy productivity 07.35

Goal 8:
43- Real GDP per capita - growth rate 08.10
44- Young people neither in employment nor in education and training 08.20
45- Total employment rate 08.30
46- Long-term unemployment rate 08.31
47- Involuntary temporary employment 08.35
48- Fatal accidents at work by sex (NACE Rev. 2, A, C-N) - Unstandardised incidence rate 08.60
Goal 9:
49 - Gross domestic expenditure on R&D 09.10
50 - Employment in high- and medium-high technology manufacturing sectors and knowledge intensive service sectors 09.11
51 - Total R&D personnel 09.13
52 - Patent applications to the European Patent Office (EPO) 09.14
53 - Share of collective transport modes in total passenger land transport 09.40
54 - Share of rail and inland waterways activity in total freight transport 09.41

Goal 10:
55 - GDP per capita in PPS 10.10
56 - Real adjusted gross disposable income of households per capita in PPS 10.11
57 - Relative median at-risk-of-poverty gap 10.22
58 - Gini coefficient of equivalised disposable income 10.24
59 - Income growth of the bottom 40 per cent of the population and the total population 10.25
60 - Number of first time asylum applications (total and accepted) per capita 10.31

Goal 11:
61 - Overcrowding rate by degree of urbanisation 11.12
62 - Distribution of population by level of difficulty in accessing public transport 11.21
63 - People killed in road accidents 11.25
64 - Urban population exposure to air pollution by particulate matter 11.31
65 - Proportion of population living in households considering that they suffer from noise 11.36
66 - Recycling rate of municipal waste 11.52

Goal 12:
67 - Generation of waste excluding major mineral wastes 12.10
68 - Recycling and landfill rate of waste excluding major mineral wastes 12.11
69 - Consumption of toxic chemicals 12.30
70 - Resource productivity 12.40
71 - Volume of freight transport relative to GDP 12.54

Goal 13:
73 - Greenhouse gas emissions (indexed totals and per capita) 13.11
74 - Greenhouse gas emissions intensity of energy consumption 13.14
75 - Global (and European) near surface average temperature 13.21
76 - Economic losses caused by climate extremes (consider climatological, hydrological, meteorological) 13.45
77 - Contribution to the 100bn international commitment on climate related expending (public finance) 13.51
78 - Share of EU population covered by the new Covenant of Mayors for Climate and Energy (integrating mitigation, adaptation, and access to clean and affordable energy) 13.63

Goal 15:
84 - Forest area as a proportion of total land area 15.11
85 - Artificial land cover per capita 15.11
86 - Change in artificial land cover per year 15.24
87 - Common bird index 15.31
88 - Sufficiency of terrestrial sites designated under the EU habitats directive 15.32
89 - Estimated soil erosion by water 15.41

Goal 17:
96 - Official development assistance as share of gross national income 17.10
97 - EU financing for developing countries 17.11
98 - EU Imports from developing countries 17.12
99 - General government gross debt 17.13
100 - Shares of environmental and labour taxes in total tax revenues 17.19
ASSESSMENT of HCI model: 17 goals

- **BIC = 2472.65**
- **Polarity:** 38 MIs need to change polarity
- **33** MIs are not statistically significant for the model (correlation \(\approx 0\))
  - (They are STATISTICS, but not INDICATORS)
- **Reliability:** 8 goals are not reliable (low Cronbach's alpha)
- **Unidimensionality:** only the goal 14 is unidimensional

100 Manifest Indicators
6 for each goal
Exploratory model: 8 factors

* 8 MIs are not statistically significant for the model

BIC= 1633.78
Thank you for your kind attention.